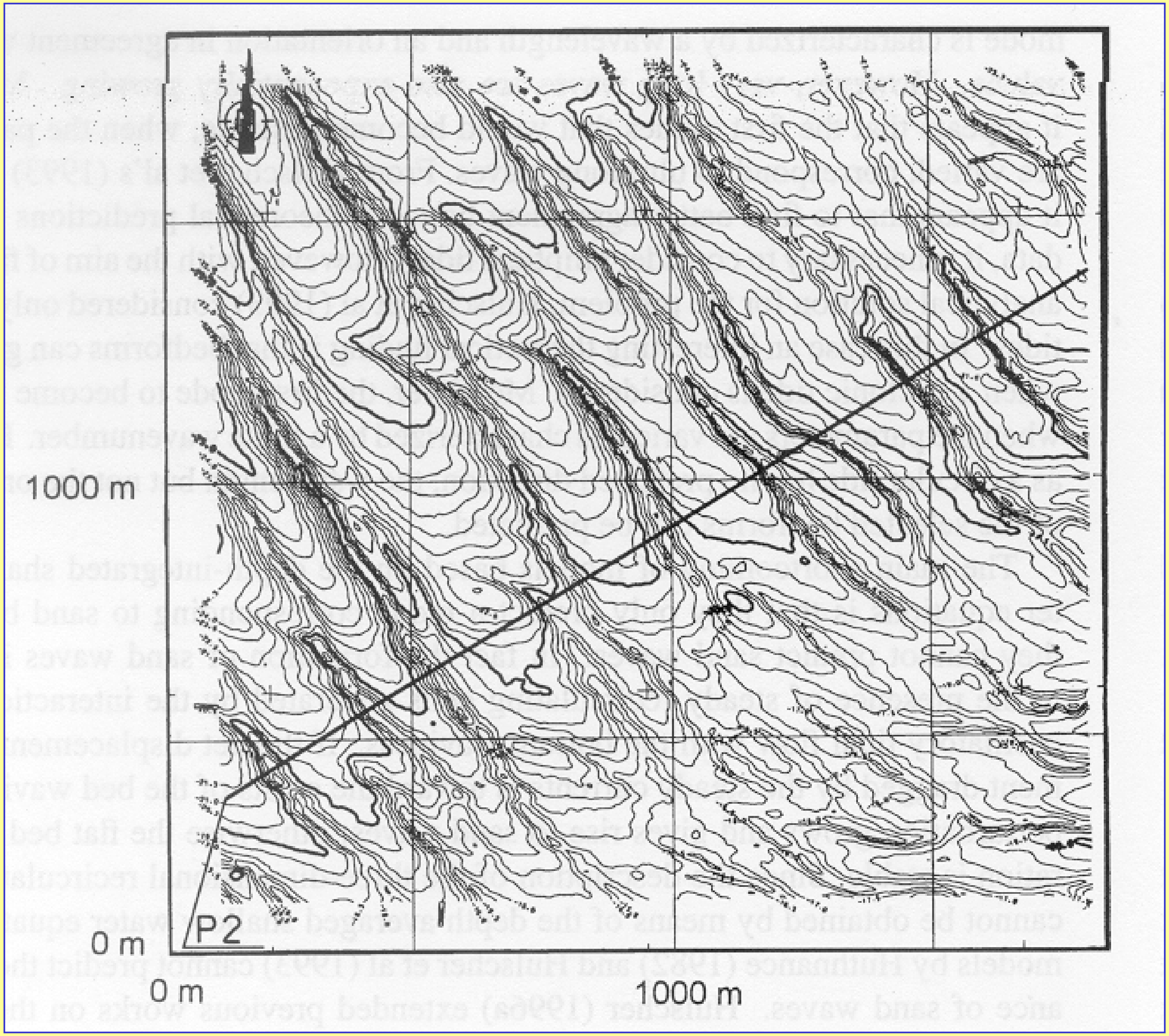
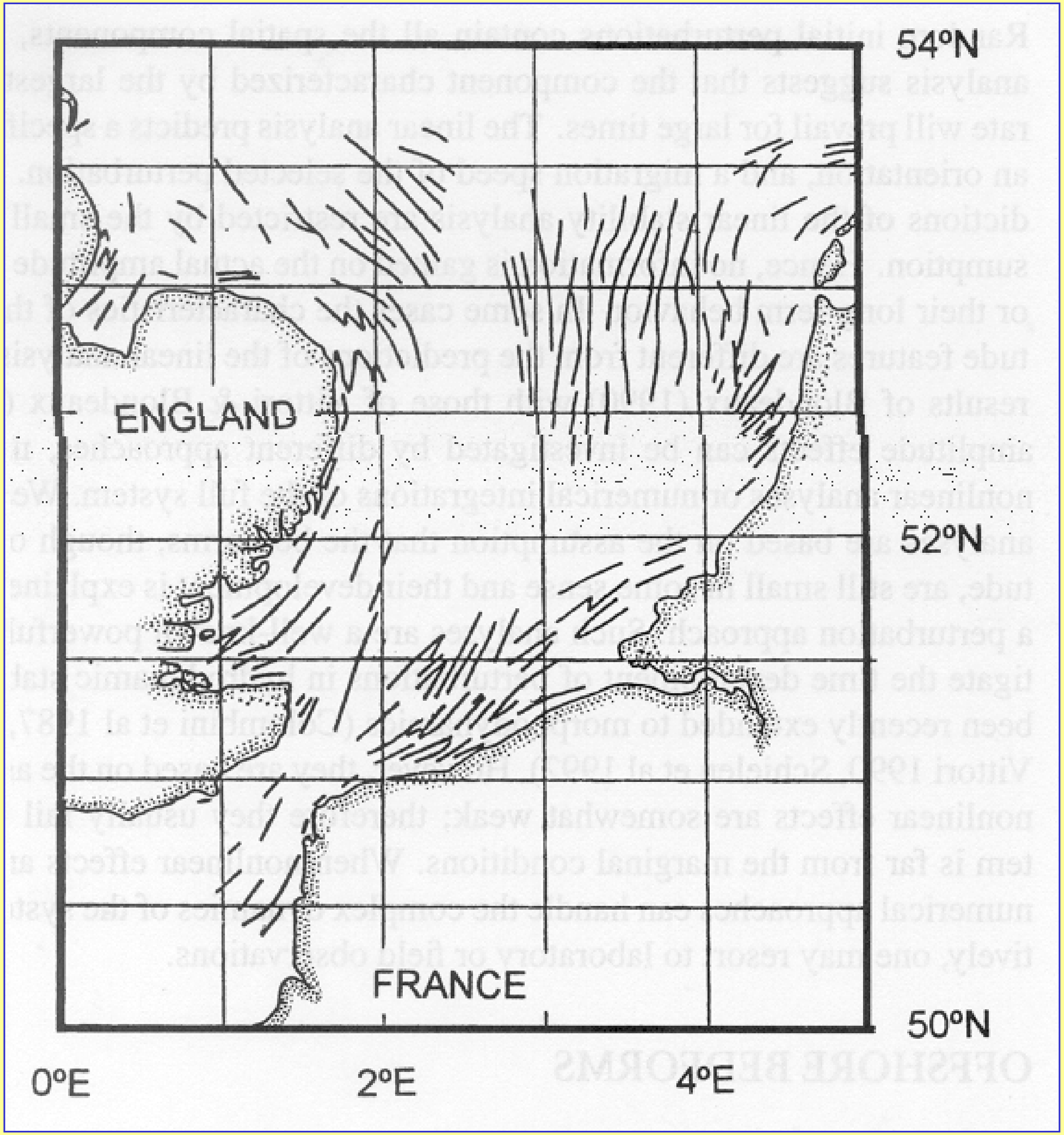
EXAMPLES OF COASTAL BEDFORMS

Ripples (O(L)=10cm)



Sand waves
(O(L)=100 m)

Bottom topography showwing the presence of sand waves (North Sea)



Sand banks (O(L)=10 km)

Sketch of the sand banks observed in the North Sea

The sea bed is full of periodic patterns such that a wavelength, a height and a migration speed can be identified

Quite often, periodic bedforms are not generated by a periodic external forcing

The process which leads to the appearance of many coastal bedforms can be explained on the basis of a stability analysis.

Therefore it is useful to present an example of a linear stability analysis

Let us consider the following problem

$$\frac{\partial U}{\partial t}+\frac{\partial U}{\partial x}\frac{\partial U}{\partial y}-\frac{\partial^2 U}{\partial y^2}+\frac{\partial^4 U}{\partial x^4}+R\frac{\partial^2 U}{\partial x^2}=0 \qquad (1)$$

with $-\infty < x < \infty$ and $0 \le y \le 1$

and the boundary conditions

$$U=0 \text{ at } y=0; \quad U=1 \text{ at } y=1 \qquad (2)$$

Now, we want to investigate the stability of this solution.

Therefore we add a perturbation to the basic solution ($U=U_0+\varepsilon U_1$) and we analyse its time behaviour.
In particular we define the energy of the perturbation

$$e(t) = \int (\varepsilon U_1)^2 \, dV \tag{3}$$

If e(t) tends to decay the solution is defined to be stable, otherwise it is unstable (ε is a parameter which is related to the amplitude of the perturbation).

The behaviour of U_1 can be found solving the following problem

$$\frac{\partial U_1}{\partial t}+\frac{\partial U_0}{\partial x}\frac{\partial U_1}{\partial y}+\frac{\partial U_1}{\partial x}\frac{\partial U_0}{\partial y}+\varepsilon\frac{\partial U_1}{\partial x}\frac{\partial U_1}{\partial y}$$

$$-\frac{\partial^2 U_1}{\partial y^2}+\frac{\partial^4 U_1}{\partial x^4}+R\frac{\partial^2 U_1}{\partial x^2}=0 \qquad (4)$$

with the boundary conditions

$$U_1=0 \quad at \quad y=0; \qquad U_1=0 \quad at \quad y=1 \qquad (5)$$

The behaviour of e(t) depends on its initial value e_0=e(0).

If e(t) decays for any value of e_0, the solution is globally stable.

If e(t) decays only for values of e_0 smaller than a particular value the solution is defined to be conditionally stable.

The global stability is difficult to be proved and usually the perturbation is assumed to be small, i.e. ε is assumed to be much smaller than 1. In other word a linear stability analysis is made

The problem can be semplified, by linearizing it

$$\frac{\partial U_1}{\partial t}+\frac{\partial U_1}{\partial x}-\frac{\partial^2 U_1}{\partial y^2}+\frac{\partial^4 U_1}{\partial x^4}+R\frac{\partial^2 U_1}{\partial x^2}=0 \qquad (6)$$

with the boundary conditions

$$U_1=0 \quad at \quad y=0; \qquad U_1=0 \quad at \quad y=1 \qquad (7)$$

Since the domain is infinite in the x direction, the solution can be expanded in the form.

$$U_1=\int F(\alpha,y,t)e^{i\alpha x}d\alpha \qquad (8)$$

By substituting (8) in (6) we obtain

$$\frac{\partial F}{\partial t} + i\alpha F - \frac{\partial^2 F}{\partial y^2} + \alpha^4 F - R\alpha^2 F = 0 \qquad (9)$$

with the boundary conditions

$$F = 0 \quad at \quad y = 0; \qquad F = 0 \quad at \quad y = 1 \qquad (10)$$

The solution can be written in the form

$$F = f(y,\alpha)e^{\Omega(\alpha)t} \quad i.e. \quad U_1 = \int f(y,\alpha)e^{\Omega(\alpha)t}e^{i\alpha x}d\alpha \qquad (11)$$

where f is given by the solution of

$$\Omega f + i\alpha f - \frac{\partial^2 f}{\partial y^2} + \alpha^4 f - R\alpha^2 f = 0 \qquad (12)$$

$$or \qquad \frac{\partial^2 f}{\partial y^2} + \left(-\Omega - i\alpha - \alpha^4 + R\alpha^2\right) f = 0 \qquad (13)$$

with the boundary conditions

$$f = 0 \quad at \quad y = 0; \qquad f = 0 \quad at \quad y = 1 \qquad (14)$$

The solution of (13) is straightforward

$$f = c_1 e^{\lambda_1 y} + c_2 e^{\lambda_2 y} \qquad (15)$$

where λ_1, λ_2 are the solution of

$$\lambda^2 = \Omega + i\alpha + \alpha^4 - R\alpha^2 \qquad (16)$$

$$\Downarrow$$

$$\lambda_1 = \sqrt{\Omega + i\alpha + \alpha^4 - R\alpha^2};$$

$$\lambda_2 = -\sqrt{\Omega + i\alpha + \alpha^4 - R\alpha^2} \qquad (17)$$

The forcing of the boundary conditions leads to

$$if \quad \Omega + i\alpha + \alpha^4 - R\alpha^2 > 0$$

$$c_1 + c_2 = 0 \qquad c_1 e^{\lambda_1} + c_2 e^{\lambda_2} = 0 \qquad (18)$$

and only the trivial solution exists

$$\text{if} \quad \Omega + i\alpha + \alpha^4 - R\alpha^2 < 0$$

$$c_2 = 0 \qquad c_1 = \sin\lambda_1 \qquad (19)$$

and a non trivial solution exist if

$$\sqrt{(\Omega + i\alpha + \alpha^4 - R\alpha^2)} = n\pi \qquad (20)$$

The condition (20) is called eigenrelation and leads to

$$-(\Omega + i\alpha + \alpha^4 - R\alpha^2) = (n\pi)^2 \qquad (21)$$

$$\Downarrow$$

$$\Omega = -i\alpha - \alpha^4 + R\alpha^2 - (n\pi)^2 \qquad (22)$$

Ω is a complex quantity, the real part of which controls the growth/decay of the perturbation while the imaginary part is related to its migration speed. Indeed the solution can be written in the form

$$U_1 = \int f(y,\alpha)\, e^{\Omega_r(\alpha)t}\, e^{i\alpha\left(x + \frac{\Omega_i}{\alpha}t\right)}\, d\alpha \qquad (23)$$

The real part of Ω is given by

$$\Omega_r = -\alpha^4 + R\alpha^2 - (n\pi)^2 \qquad (24)$$

$$\text{if} \quad \Omega_r = -\alpha^4 + R\alpha^2 - (n\pi)^2 > 0 \qquad (25)$$

the perturbation grows

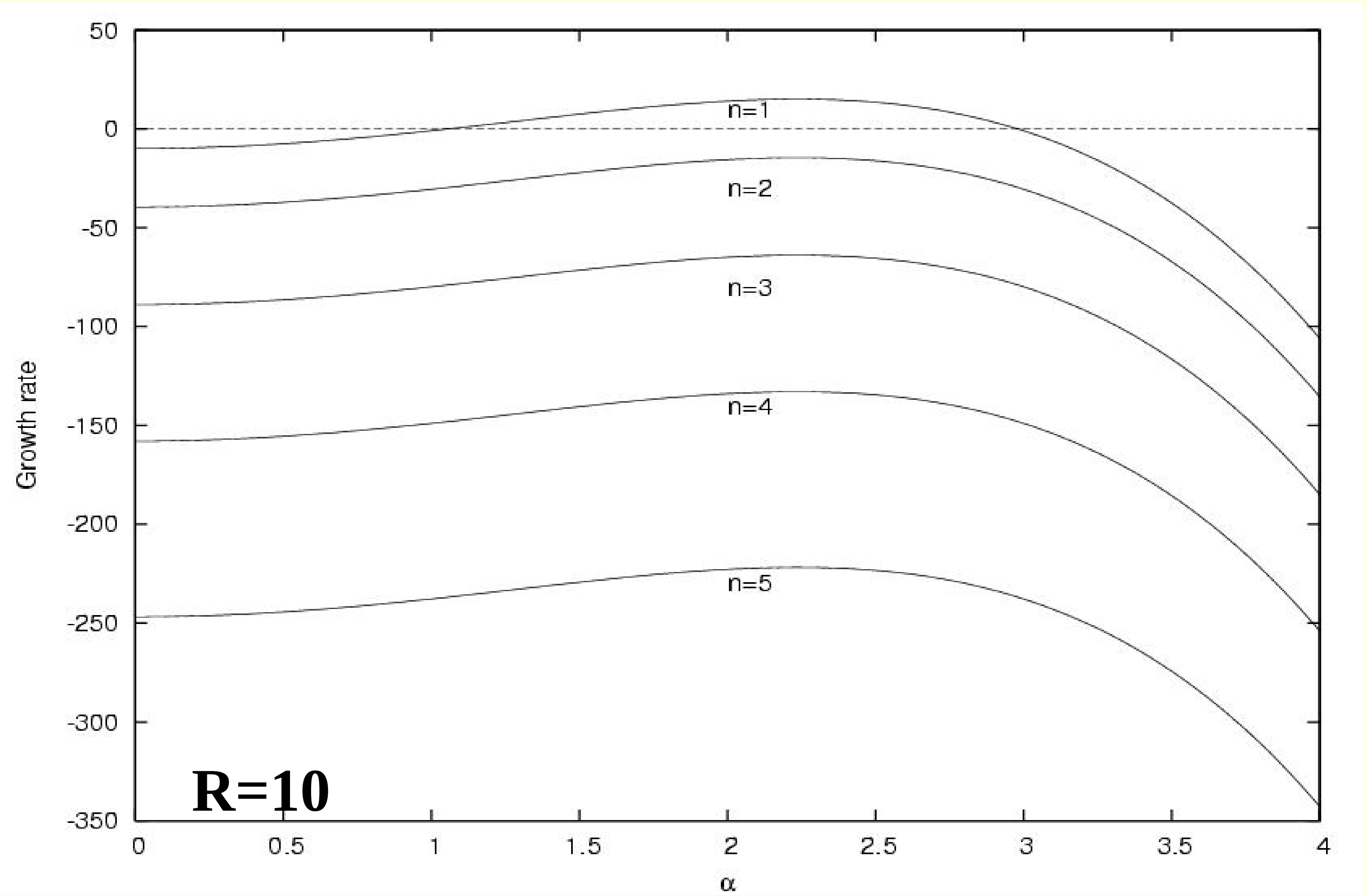
$$\text{if} \quad \Omega_r = -\alpha^4 + R\alpha^2 - (n\pi)^2 < 0 \qquad (26)$$

the perturbation decays

Let us look at the behaviour of Ω_r as function of *α* for different values of the parameter R and of the integer *n* which identifies the so called ‘mode’.
First of all, we notice that

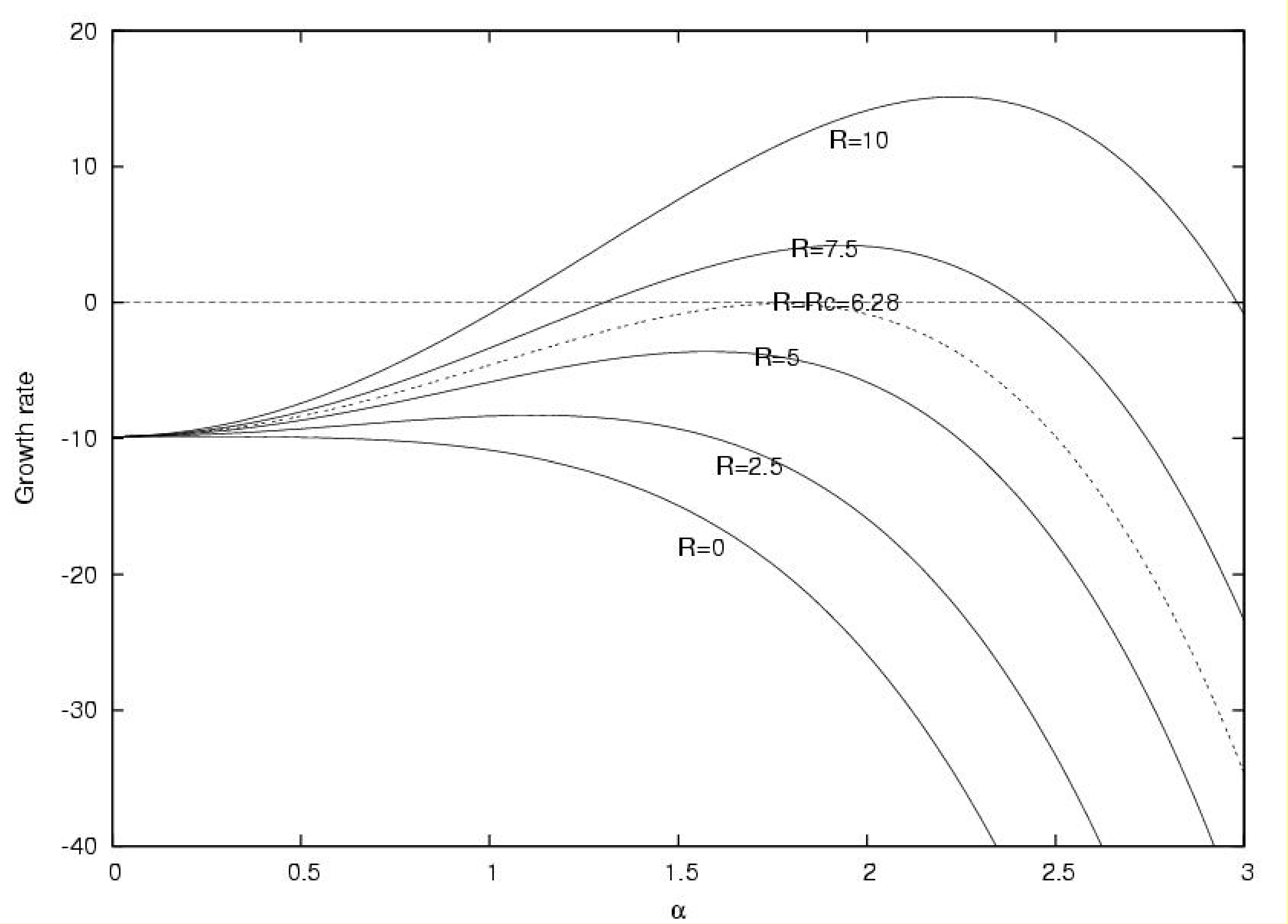
$$\frac{\partial \Omega_r}{\partial n} = -2n\pi^2 \qquad (27)$$

Therefore the growth rate Ω_r becomes smaller if n is increased.



If we want to look for the values of the paramenters which give rise to positive values of Ω_r it appears convenient to fix n=1

Growth rate Ω_r for n=1 and different values of R



For R=0 the growth rate is negative for any value of α. Increasing R, a value R_c is found, such that for $\alpha=\alpha_c$ the growth rate vanishes. Then larger values of R give rise to positive values of the growth rate for α falling in a finite interval. The evaluation of α_c and R_c is straightforward. First we notice that the maximum value

The maximum growth rate Ω_r, for fixed values of R and n, is

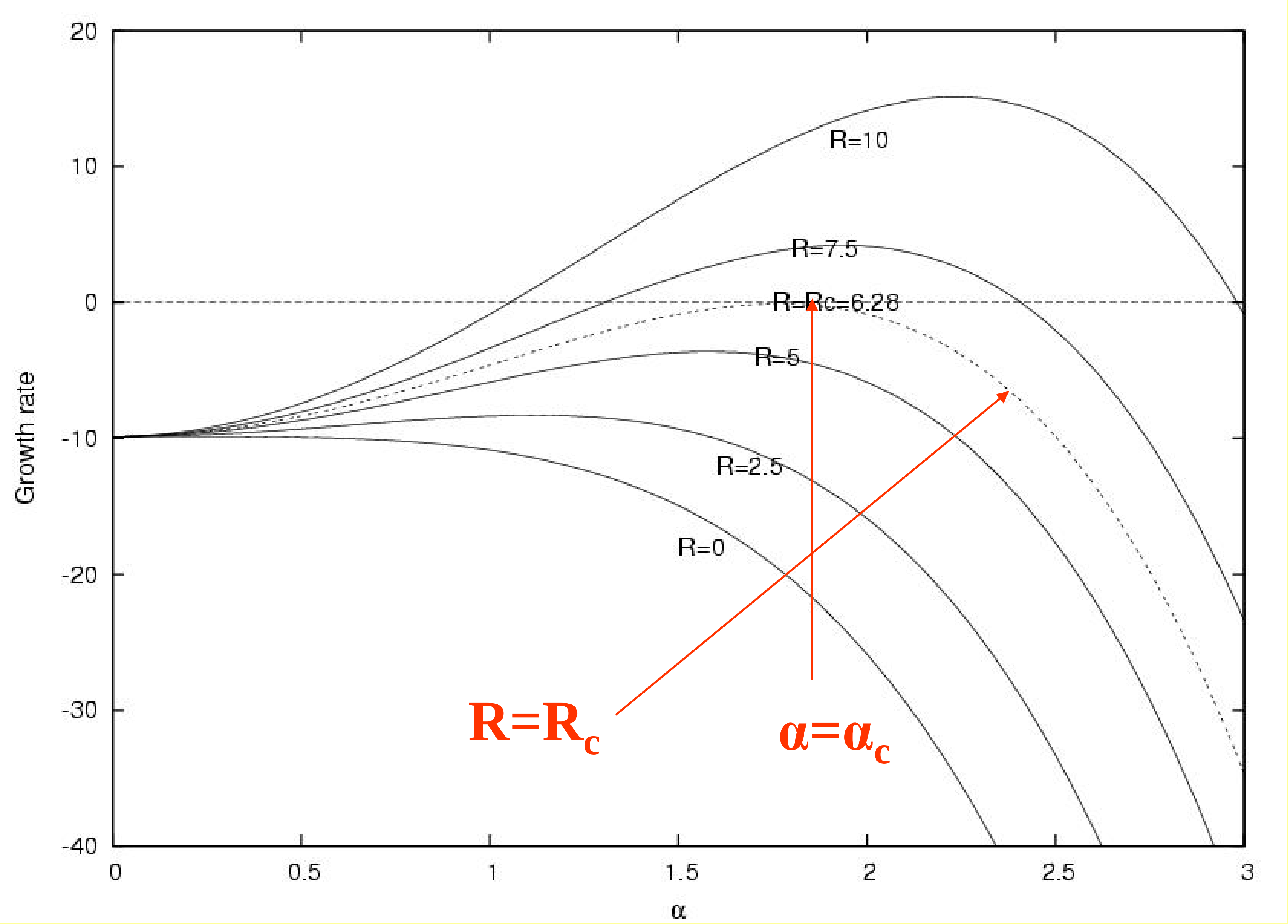
$$\Omega_r\left(\sqrt{\frac{R}{2}}\right) = -(n\pi)^2 + \frac{R^2}{4} \qquad (30)$$

Therefore the maximum value of the growth rate vanishes when

$$R = R_c = 2n\pi$$

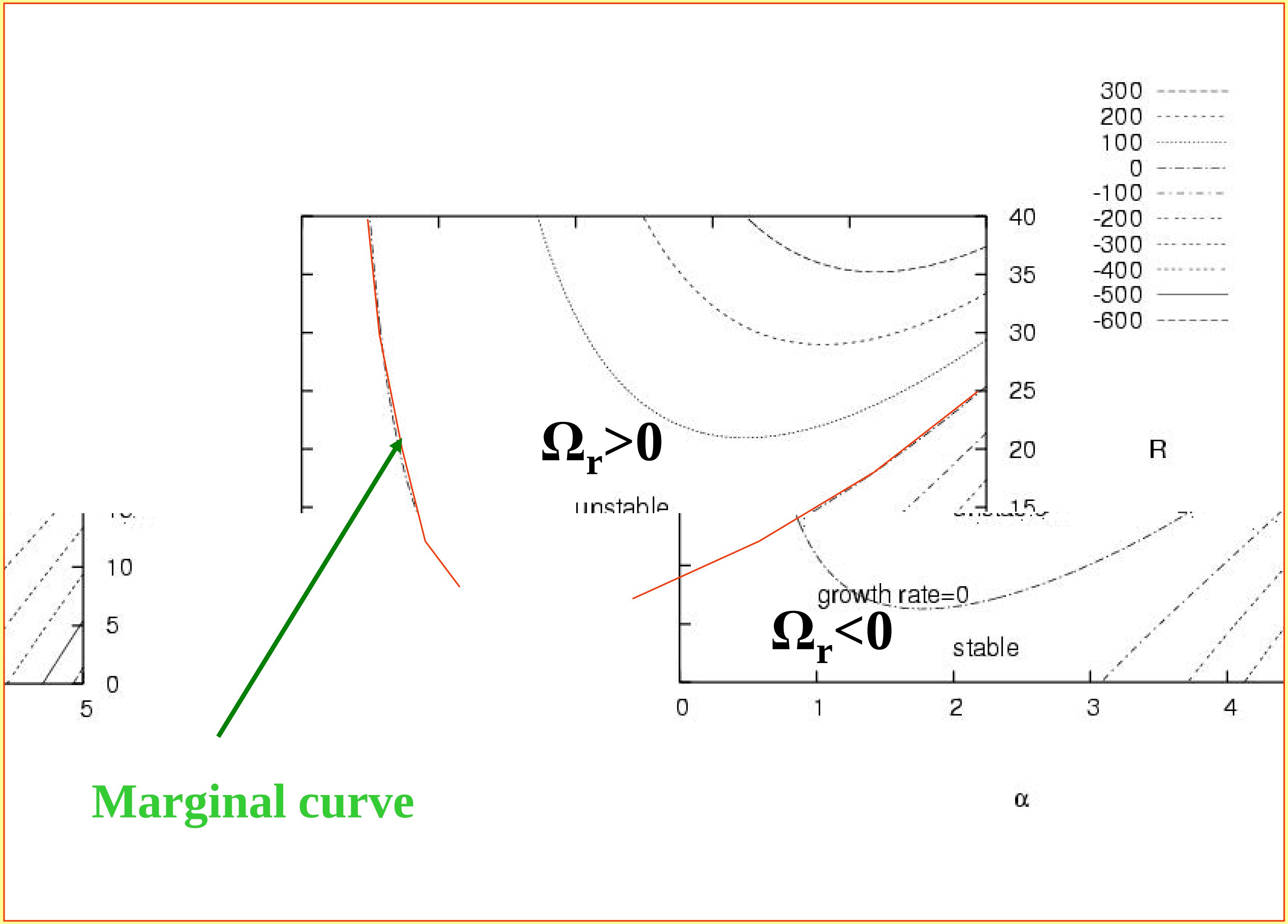
$$(31)$$

$$\alpha = \alpha_c = \sqrt{n\pi}$$

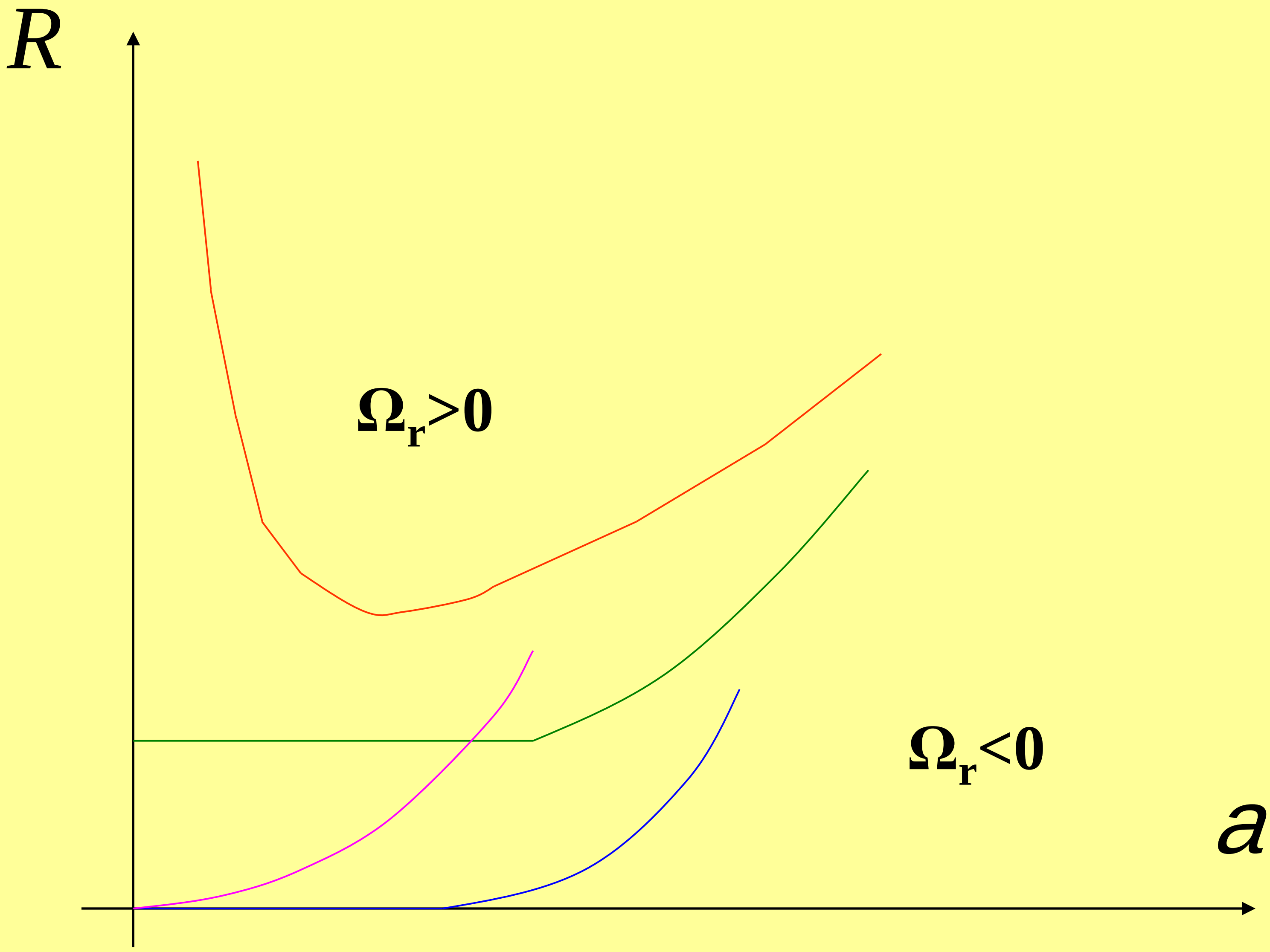


R=10
R=7.5
R=Rc=6.28
R=5
R=2.5
R=0
R=R_c
α=α_c
Growth rate
α
20
10
0
-10
-20
-30
-40
0
0.5
1
1.5
2
2.5
3

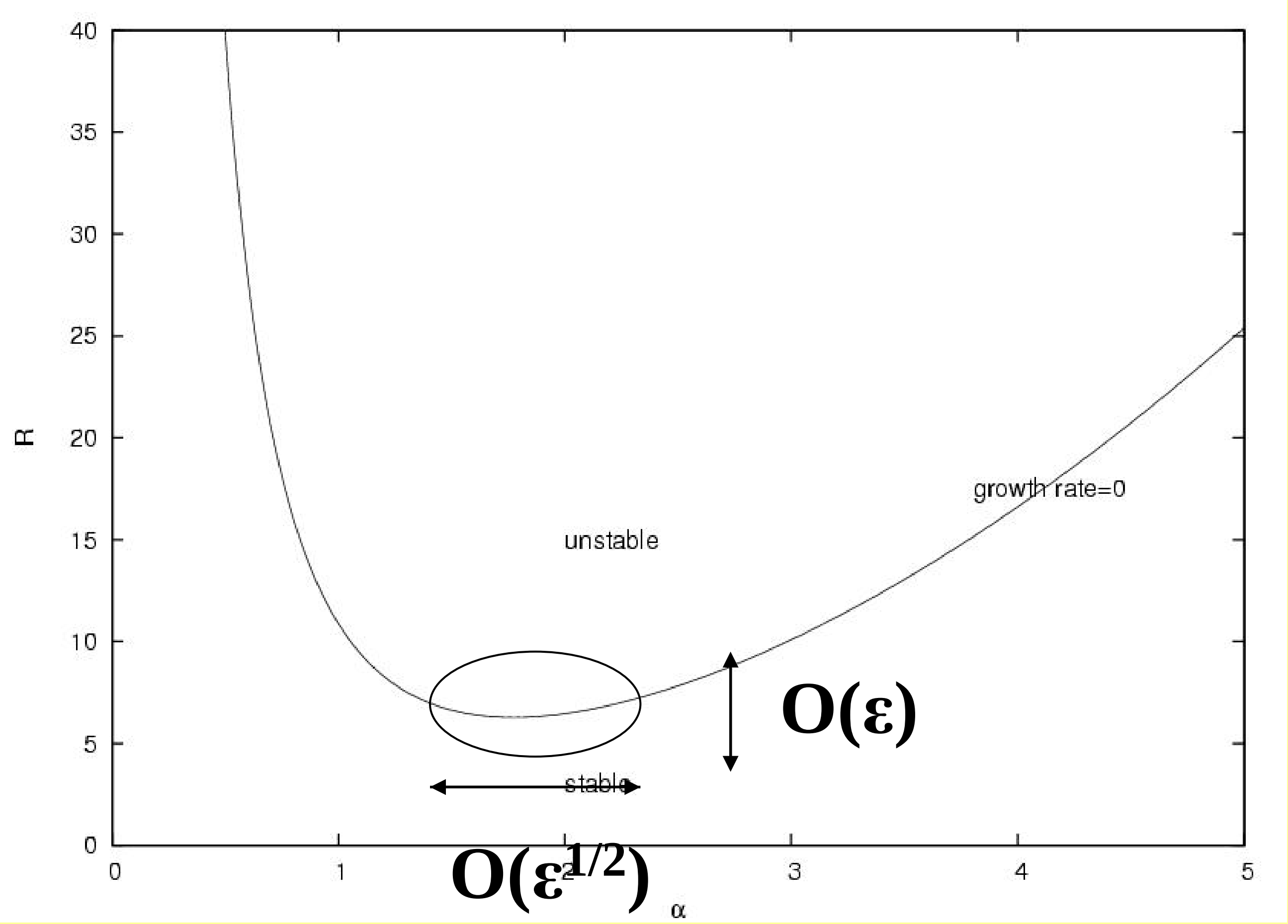
Growth rate versus α and R



Possible marginal curves



Marginal stability curve



Let us consider a small range of α and R around α_c and R_c such that α-α_c is of order $\varepsilon^{1/2}$ and R-R_c is of order ε

$$R = R_c + \varepsilon R_1 \qquad (32)$$

In this range we can compute the growth rate by means of Taylor expansion

$$\Omega_r = \Omega_r(\alpha_c, R_c) + \left.\frac{\partial \Omega_r}{\partial \alpha}\right|_{\alpha_c, R_c} (\alpha - \alpha_c) \qquad (33)$$

$$+ \left.\frac{\partial \Omega_r}{\partial R}\right|_{\alpha_c, R_c} (R - R_c) + \frac{1}{2} \left.\frac{\partial^2 \Omega_r}{\partial \alpha^2}\right|_{\alpha_c, R_c} (\alpha - \alpha_c)^2 + \dots$$

The first two terms on the right hand side vanish. Therefore the growth rate turns out to be of order ε. Hence it reasonable to assume that the growth of the perturbation takes place on the slow temporal scale $\tau=\varepsilon t$. Moreover, the different perturbations component interact each with the other producing modulations which take place on the slow spatial scale $\xi=\varepsilon^{1/2}x$

In the small range around the critical conditions we assume that the instability of the basic solution leads to the growth of the mode n=1 characterized by a wavenumber α_c and an amplitude which depends on

$$\tau = \varepsilon t \qquad\qquad \xi = \varepsilon^{1/2} x \qquad\qquad (34_{a,b})$$

Let us introduce these the new temporal and spatial variables such that

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau} \qquad\qquad \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial x} + \varepsilon^{1/2} \frac{\partial}{\partial \xi} \qquad\qquad (35_{a,b})$$

Moreover let us assume that the perturbation has an unknown amplitude of order ε^{z}

$$U_1 = \varepsilon^{z} A(\tau, \xi) \sin(\pi y) e^{i\alpha_c x} e^{i\Omega_i t} + c.c. + \varepsilon^{2z}(\ldots \qquad\qquad (36)$$

By plugging (36) into (6) and taking into account (32), at the leading order of approximation (O(ε^z)) the problem is satisfied. Let us consider the problem at order ε^{2z}. The structure of the forcing terms suggests that

$$
\begin{aligned}
U_1 &= \varepsilon^{z} A(t,x)\sin(\pi y)e^{i(\alpha_c x+\Omega_i t)} + c.c. \qquad (37)\\
&\quad + \varepsilon^{2z}\left[A^2 f_{22}(y)e^{i(\alpha_c x+\Omega_i t)} + A\overline{A}f_{20} + c.c.\right] + O(\varepsilon^{3z})
\end{aligned}
$$

By plugging (37) into the problem, at order ε^{2z} we obtain the following problems for f_{22} and f_{20}

$$\frac{d^2 f_{22}}{dy^2} - 8p^2 f_{22} = \frac{ip\sqrt{p}}{2}\sin(2py)$$

$$\frac{d^2 f_{20}}{dy^2} = -\frac{ip\sqrt{p}}{2}\sin(2py)$$

with homogenous boundary conditions. It follows

$$f_{22} = -\frac{i}{24\sqrt{p}}\sin(2py)$$

$$f_{20} = \frac{i}{8\sqrt{p}}\sin(2py)$$

Then, at the following order of approximation we have the following forcing terms

$$\dots = -\varepsilon^{3z} A^2 \overline{A} \sin(\pi y) e^{i(\alpha_c x + \Omega_i t)} \left[\frac{\alpha_c \pi}{8\sqrt{\pi}} + \frac{\alpha_c \pi}{24\sqrt{\pi}} + \frac{\alpha_c \pi}{24\sqrt{\pi}} \right]$$

$$- \varepsilon^{z+1} \sin(\pi y) e^{i(\alpha_c x + \Omega_i t)} \left[\frac{\partial A}{\partial t} - \pi R_1 A \right] + \dots$$

In order to find a solution a solvability condition should be forced which implies that

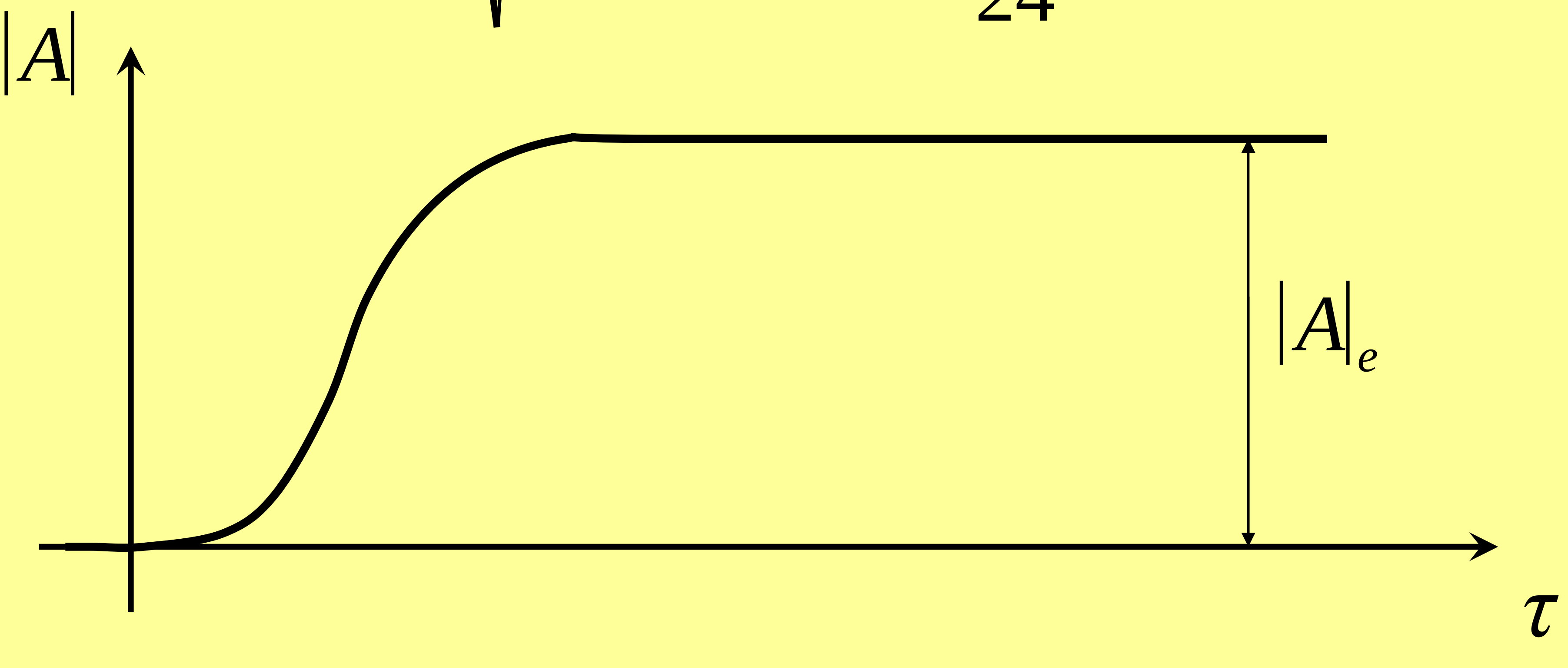
1) 3z must be equal to z+1 and therefore z=1/2;

2)

$$\frac{\partial A}{\partial t} - \pi R_1 A + \frac{5}{24} \pi A^2 \overline{A} + \dots = 0$$

If the spatial modulation of the most unstable mode is neglected the amplitude equation reduces to the so-called Stuart-Landau equation, the solution of which is

$$|A| = \sqrt{\frac{\pi R_1}{\exp[-2\pi R_1 \tau] + \frac{5\pi}{24}}}$$



For τ tending to infinity, the solution tends to the so-called equilibrim amplitude which can be obtained simply by forcing dA/dτ=0 and turns out to be

$$|A_e| = \sqrt{\frac{24R_1}{5}}$$

THE END !!!!!!!